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Engineering Notes

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Optimal Missions with Minimagnetospheric Plasma Propulsion

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Introduction

THE physical principle of minimagnetospheric plasma propulsion (M2P2) is to create a magnetic bubble around the spacecraft and to use the deflection of the ambient plasma to obtain momentum and increase the spacecraft thrust.[1] In a recent paper,[2] Trask et al. investigate the convenience of using M2P2 in addition to a conventional high-thrust propulsion system, to minimize the propelling mass necessary for an interplanetary rendezvous mission. When a two-dimensional transfer problem between circular and coplanar orbits is assumed, an analytical procedure is derived for estimating the time of flight during the M2P2 thrust period. The authors employ a pure radial trust and neglect the propellant consumption due to the M2P2. Using a direct approach, they[2] also investigate the effects on the mission performance of an additional offaxis thrust, obtained by rotating by a small angle the thrust vector in the orbital plane. The capability of providing a small azimuthal thrust renders the M2P2 behavior similar to that of a solar sail. On the other hand, there are two substantial differences between these two propulsion systems: the mass variation associated with the M2P2 propellant consumption and the magnitude of the azimuthal acceleration. In fact, for a correct operation of the M2P2 system, the maximum azimuthal thrust angle must be smaller than approximately 5 deg (Ref. 1). Nevertheless, the advantage of the M2P2 system is that the deployment occurs through electromagnetic processes and does not require large mechanical structures.

In their analysis,[2] Trask et al. assume that the thrust vector modulus remains constant during the rotation. However, this is an approximation because, inasmuch as the thrust vector rotates in the orbital plane, the plasma-inflated magnetic bubble increases along with the net thrust,[1] as better explained later.

The aim of this Note is to develop a more refined thrust model for the description of a M2P2 system that takes into account both the radial thrust variation due to an azimuthal thrust component and the corresponding increase in propellant consumption. In doing so, a semi-analytical model is proposed where the radial thrust and the mass flow rate depend on the azimuthal thrust component through a polynomial approximation. Also, the optimal control law minimizing the velocity change required to circularize the final orbit is found using an indirect method. The effectiveness of the control law is demonstrated by simulating interplanetary rendezvous missions to Mars and Jupiter.

Mathematical Model

The heliocentric equations of motion for a spacecraft in a polar inertial frame $\mathcal{T}_\odot(r, \theta)$ are

$$\dot{r} = u \tag{1}$$

$$\dot{\theta} = v/r \tag{2}$$

$$\dot{u} = v^2/r - \mu_\odot/r^2 + \tau F_r/m \tag{3}$$

$$\dot{v} = -uv/r + \tau\zeta|F_\theta|/m \tag{4}$$

$$\dot{m} = -\tau\beta \tag{5}$$

where $\mu_\odot$ is the sun's gravitational parameter, r is the heliocentric distance of the spacecraft from the sun, θ is the polar angle measured anticlockwise from some reference position, u and v are the radial and azimuthal velocities, $F_r \geq 0$ and F_θ denote radial and azimuthal propulsive thrusts, m is the spacecraft mass, and β is the mass flow rate. Also, $\tau = (0, 1)$ models the thruster on/off condition and $\zeta = \text{sgn}(F_\theta)$, where sgn $(\cdot)$ is the signum function.

Assume that the M2P2 dipole axis can be tilted by a prescribed angle from the radial direction to produce a net azimuthal thrust F_θ on the spacecraft.[1] This additional thrust is the result of an increase of the plasma-inflated magnetic bubble due to the stronger magnetic fields at the poles. As a result, the magnetic bubble is capable of intercepting more thrust from the solar wind because it has an higher energy state. The shape deformation of the magnetic bubble is, in turn, responsible for an increase in both the mass flow rate β (because some additional propellant needs to be expended to maintain the dipole axis orientation) and the radial thrust F_r. For further in-depth treatment of M2P2 characteristics, the reader is referred to the original paper by Winglee et al.[1]

Because the tilting of the dipole axis is ultimately connected to the value of the azimuthal thrust magnitude, for mathematical convenience both the variations in β and F_r are modeled as a function of $|F_\theta|$ through second-order polynomial approximations, that is,

$$F_r = F_{r_0}\left[1 + b_1\left(|F_\theta|/F_{r_0}\right) + b_2\left(|F_\theta|/F_{r_0}\right)^2\right] \tag{6}$$

$$\beta = \beta_0\left[1 + b_3\left(|F_\theta|/F_{r_0}\right) + b_4\left(|F_\theta|/F_{r_0}\right)^2\right] \tag{7}$$

where $|F_\theta| \in [0, |F_\theta|_{\max}]$. Note that $F_{r_0} > 0$ and $\beta_0 > 0$ represent the values of mass flow rate and radial thrust when the dipole axis is orthogonal to the solar wind and the corresponding azimuthal thrust is zero. In Eqs. (6) and (7), the constants $b_1, \ldots, b_4$ represent dimensionless coefficients that are obtained through a best-fit approximation of experimental and/or numerical data. When Eqs. (6) and (7) are substituted into Eqs. (1–5), the equations of motion can be written in compact form as

$$\dot{\boldsymbol{x}} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}) \tag{8}$$

where $\boldsymbol{x} = [r, \theta, u, v, m]^T$ is the state vector and $\boldsymbol{u} = [\zeta, \tau, |F_\theta|]^T$ is the control vector.

Assume we are given a starting initial $\boldsymbol{x}_0$ state. We look for the optimal control law $\boldsymbol{u}(t)$ that minimizes the impulsive velocity change

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ΔV required to circularize the orbit at a prescribed final distance r_f. Equivalently, the performance index

$$J = -(\Delta V)^2 = -u_f^2 - \left(v_f - \sqrt{\mu_\odot / r_f}\right)^2 \tag{9}$$

should be maximized, where u_f and v_f are the spacecraft radial and azimuthal velocity at the unspecified terminal time t_f. From Eqs. (1–5), the Hamiltonian associated with the problem is

$$\begin{aligned} H = {} & \lambda_r u + \lambda_\theta (v/r) + \lambda_u \left(v^2/r - \mu/r^2 + \tau F_r/m\right) \\ & + \lambda_v(-uv/r + \tau\zeta|F_\theta|/m) - \lambda_m \tau\beta \end{aligned} \tag{10}$$

where $\lambda_r, \lambda_\theta, \lambda_u, \lambda_v$, and λ_m are the adjoint variables associated with the state variables r, θ, u, v, and m, respectively. The time derivatives of the adjoint variables are provided by the Euler–Lagrange equations,

$$\dot{\lambda}_r = \lambda_\theta v \big/ r^2 + \lambda_u (v^2/r^2 - 2\mu/r^3) - \lambda_v (uv/r^2) \tag{11}$$

$$\dot{\lambda}_\theta = 0 \tag{12}$$

$$\dot{\lambda}_u = -\lambda_r + \lambda_v (v/r) \tag{13}$$

$$\dot{\lambda}_v = -\lambda_\theta/r - 2(\lambda_u v/r) + \lambda_v u/r \tag{14}$$

$$\dot{\lambda}_m = \lambda_u \tau F_r \big/ m^2 + \lambda_v \tau\zeta|F_\theta| \big/ m^2 \tag{15}$$

From Pontryagin's maximum principle, the optimal control law must maximize the Hamiltonian at any time. Consider first the problem of maximizing H with respect to ζ and τ. Because the Hamiltonian is linear in both ζ and τ, the optimal control strategy is simply given by

$$\zeta = \operatorname{sgn}(\lambda_v) \tag{16}$$

$$\tau = [\operatorname{sgn}(\lambda_u F_r + \lambda_v \zeta |F_\theta| - m\lambda_m\beta) + 1]/2 \tag{17}$$

Next, when it is assumed that the thruster is on ($\tau = 1$), the optimal $|F_\theta|$ can be obtained by maximizing that portion H' of the Hamiltonian that explicitly depends on the control vector. From Eqs. (6) and (7), it turns out that H' is a quadratic function of $|F_\theta|$, that is,

$$H' \triangleq \mathcal{A} F_\theta^2 + \mathcal{B}|F_\theta| + \mathcal{C} \tag{18}$$

where

$$\mathcal{A} \triangleq \frac{\lambda_u F_{r_0} b_2 - \lambda_m m \beta_0 b_4}{F_{r_0}^2 m} \tag{19}$$

$$\mathcal{B} \triangleq \frac{\lambda_u F_{r_0} b_1 + \lambda_v F_{r_0} \zeta - \lambda_m m \beta_0 b_3}{F_{r_0} m} \tag{20}$$

$$\mathcal{C} \triangleq \frac{\lambda_u F_{r_0} - \lambda_m m \beta_0}{m} \tag{21}$$

Accordingly, the optimal control law is given by

$$|F_\theta| = \begin{cases} |F_\theta|_{\max} \times \dfrac{[\operatorname{sgn}(|F_\theta|_{\max} + \mathcal{B}/\mathcal{A}) + 1]}{2}, & \text{if } \mathcal{A} > 0 \\ 0, & \text{if } \mathcal{A} < 0 \cap \mathcal{B} < 0 \\ |F_\theta|_{\max}, & \text{if } \mathcal{A} < 0 \cap -\dfrac{\mathcal{B}}{2\mathcal{A}} > |F_\theta|_{\max} \\ -\dfrac{\mathcal{B}}{2\mathcal{A}}, & \text{if } \mathcal{A} < 0 \cap -\dfrac{\mathcal{B}}{2\mathcal{A}} \in [0, |F_\theta|_{\max}] \end{cases} \tag{22}$$

There are some special cases that deserve comment. First note that if $\mathcal{A} = 0$, then H' is a linear function of $|F_\theta|$ and the optimal control law reduces to

$$|F_\theta| = |F_\theta|_{\max}[\operatorname{sgn}(\mathcal{B}) + 1]/2[-4pt] \tag{23}$$

Clearly, Eq. (23) is also valid as long as the condition $b_2 = b_4 = 0$ is met, that is, when both F_r and β vary linearly with $|F_\theta|$ [see Eqs. (6) and (7)]. Also note that the condition $|F_\theta|_{\max} = 0$ corresponds to a pure radial thrust and a constant mass flow rate. This case coincides with the problem analyzed by Trask et al.[2] The corresponding optimal control law is the simple on/off law given by Eq. (17) setting $|F_\theta| \equiv 0$.

Another interesting problem is obtained assuming an azimuthal thrust with constant modulus, which corresponds to keeping a fixed value of both the dipole tilt angle and the thrust angle. In this case, the control law is given by Eqs. (16) and (17).

To complete the problem setup, assume that at the initial time t_0 the spacecraft with mass m_0 is in a circular orbit of radius r_0 and that the final polar angle $\theta(t_f)$ is left free. The boundary conditions of the differential problems (1–5) and (11–15) are given by

$$\begin{aligned} r(t_0) = r_0, &\qquad \theta(t_0) = u(t_0) = 0 \\ v(t_0) = \sqrt{\mu_\odot / r_0}, &\qquad m(t_0) = m_0 \end{aligned} \tag{24}$$

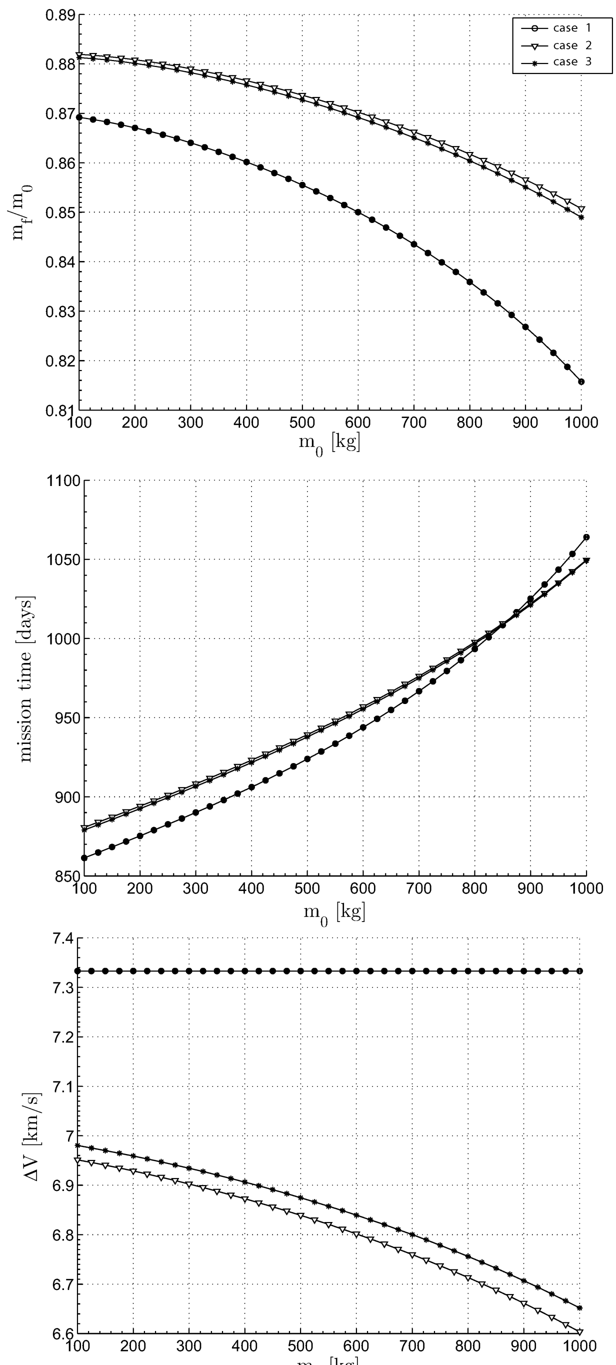

Fig. 1 Earth–Jupiter mission.

$$r(t_f) = r_f, \qquad \lambda_u(t_f) = -2\,u(t_f)$$
$$\lambda_v(t_f) = -2\left(v_f - \sqrt{\mu_\odot/r_f}\right), \qquad \lambda_m(t_f) = \lambda_\theta(t_f) = 0 \quad (25)$$

The optimal mission time is obtained by enforcing the transversality condition $H(t_f) = 0$ (Ref. 3).

Numerical Simulations

For illustrative purposes, Earth–Mars and Earth–Jupiter missions have been studied. The target orbit radii in astronomical units (AU) are 1.5237 AU for Mars orbit and 5.2028 AU for Jupiter orbit. A set of canonical units[4] have been used in the integration of the differential equations to reduce their numerical sensitivity. The differential equations have been integrated in double precision using a Runge–Kutta fifth-order scheme with absolute and relative errors of 10^{-10}. The boundary-value problem associated to the variational problem has been solved through a hybrid numerical technique combining genetic algorithms (to obtain an estimate of the adjoint variables), with gradient-based and direct methods to refine the solution.[5]

Different scenarios have been investigated. For comparative purposes with Trask et al.[2] two simplified problems (cases 1 and 2) have been first simulated. Case 1 corresponds to a pure radial thrust $F_r = F_0$, whereas in case 2 we set $F_r = F_0 \cos(\pi/36)$ and $|F_\theta| = F_0 \sin(\pi/36)$. In both cases a constant thrust $F_0 = 1$ N and a constant mass flow rate $\beta = 0.5$ kg/day are assumed.[2] Note that case 2 corresponds to a 5-deg in-plane thrust angle from the radial direction. Finally, the optimal control laws already described have been applied to study the problem of transfer to Mars and Jupiter (case 3). In doing so, the physical thruster characteristics have been derived from Winglee et al.[1] In particular, when $\theta = 0$, we assume $F_\theta = 0$ and $F_r = F_{r_0} = 0.3$ N, whereas for $\theta = \theta_{\max}$, we take $F_\theta = F_{\theta_{\max}} = 0.08$ N and $F_r = 1$ N. The corresponding coefficients in Eq. (6) are $b_1 = 0$ and $b_2 = 33$. Also, because of a lack of specific information about the variations of β, a constant mass flow rate $\beta = 0.5$ kg/day is assumed in Eq. (7).

Earth–Mars and Earth–Jupiter missions have been simulated with different values of the initial mass m_0 in the range [100, 1000] kg. This parametric study allows one to investigate the impact of the spacecraft mass on the mission characteristics. Because both missions show similar trends, only Earth–Jupiter data are shown and summarized in Fig. 1.

Note that the simulation outputs corresponding to $m_0 = 100$ kg for both cases 1 and 2 coincide with the solutions found by Trask et al.[2] The mission times increase with m_0 due to a corresponding reduction of spacecraft acceleration. For small values of the spacecraft mass, a pure radial thrust (case 1) allows one to obtain better performance in terms of mission time at the expense, of course, of a substantial increase in the impulsive velocity change ΔV required to circularize the orbit. However, because the slope of the mission time vs m_0 is different for the three cases, there is a value of m_0 beyond which a pure radial thrust makes the time of flight increase. Also, for cases 2 and 3, the total ΔV decreases as m_0 increases. Finally, assuming an initial mass $m_0 = 100$ kg, the corresponding trajectory is shown in Fig. 2 for case 3. The thruster is on in the first phase of the transfer mission only, then, a coast arc follows whose apoapses coincides with the Jupiter orbit. The thrust strategy is similar also for both cases 1 and 2 and confirms the results found by Trask et al.[6]

Conclusions

A semi-analytical model has been proposed for the description of an M2P2 system. This model takes into account both the radial thrust variations due to the presence of an azimuthal thrust component and

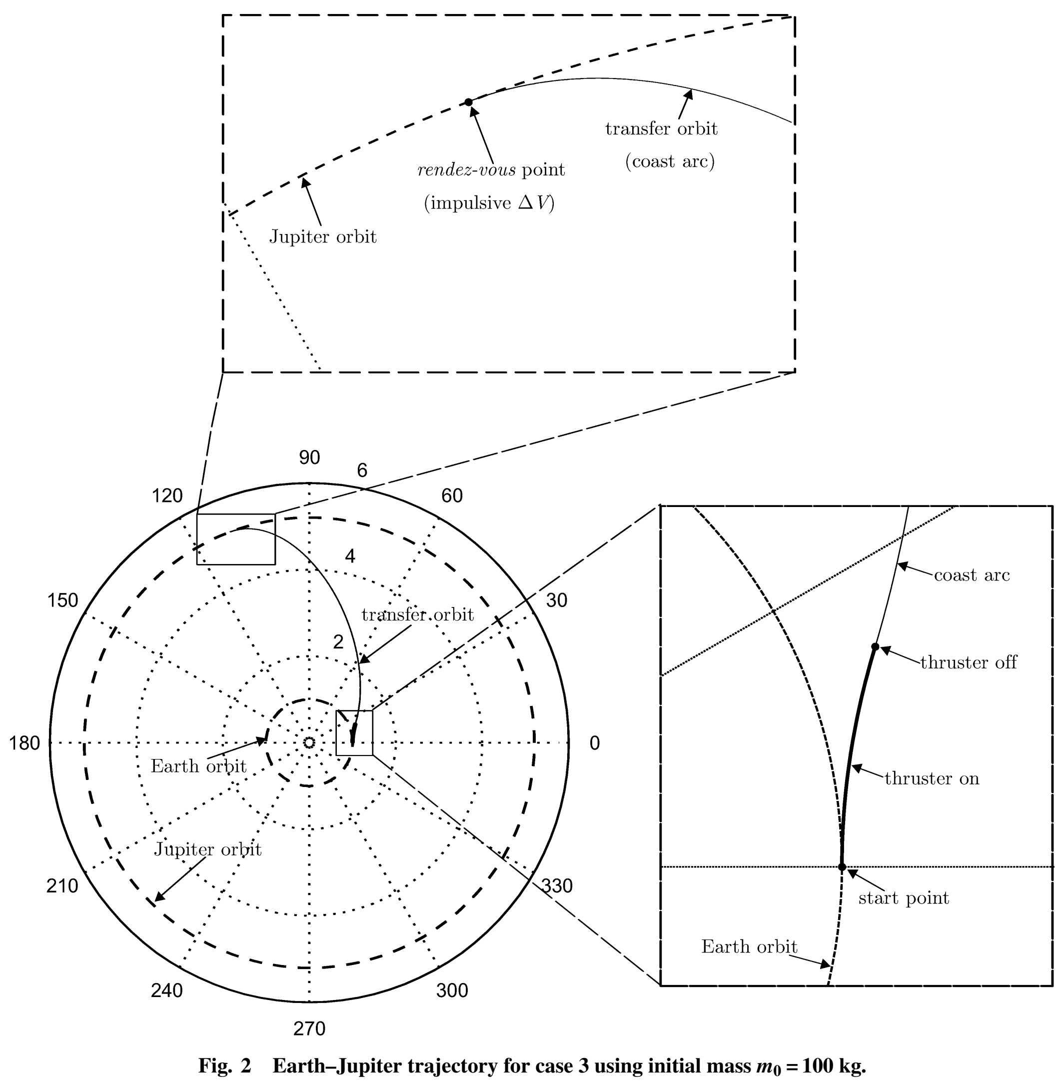

Fig. 2 Earth–Jupiter trajectory for case 3 using initial mass $m_0 = 100$ kg.

the corresponding increase in propelling consumption. The radial thrust of the system and the mass flow rate have been assumed to depend on the azimuthal thrust component through a polynomial approximation. The optimal control law minimizing the velocity change required to circularize the final orbit has been derived using an indirect method. A parametric study with different values of the spacecraft mass has been discussed to point out the impact of the mass budget on the mission characteristics. The effectiveness of the control law has been shown by simulating interplanetary rendezvous missions to Mars and Jupiter. Although the M2P2 still requires more refined theoretical and experimental studies, it appears to be a promising option for new interplanetary missions.

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